

Lévy-driven Fluid Queue with Server Breakdowns and Vacations

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Abstract

In this paper, we consider a Lévy-driven fluid queueing system where the server may subject to breakdowns and repairs. In addition, the server will leave for a vacation each time when he finds an empty system. We cast the queueing process as a Lévy process modified to have random jumps at two classes of stopping times. By using the Kella-Whitt martingale method, we obtain the limiting distribution of the virtual waiting time process. Moreover, we investigate the busy period, the correlation structure and the stochastic decomposition properties. These results may be generalized to Lévy processes with multi-class jump inputs or Lévy-driven queues with multiple input classes.

Keywords: Lévy processes; Fluid queues; Server breakdowns and vacations; Kella-Whitt martingale; Stochastic decomposition

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1. Introduction

The class of Lévy processes consisting of all stochastic processes with stationary independent increments is one of the most important family of stochastic processes arising in many areas of applied probability. It covers well-studied processes such as Brownian motion and compound Poisson processes. Lévy processes are used as models in the study of queueing systems, insurance risks, storage systems, mathematical finance and so on. For a comprehensive and outstanding analysis of Lévy processes and their applications, readers can refer to the books by Bertoin [4], Sato [22], Applebaum [1], and Kyprianou [18].

Over recent years it has been a rapid growth in the literature on queues with server breakdowns and vacations due to their widely applications in computer communication networks and manufacturing systems. Li et al. [19] considered an $M/G/1$ queue with Bernoulli vacations and server breakdowns using a supplementary variable method. Gray et al. [12] analyzed a multiple-vacation $M/M/1$ queueing model, where the service station is subject to breakdown while in operation. Ke [15] studied the control policy in $M^{[X]}/M/1$ queue with server breakdowns and multiple vacations. Jain and Jain [14] dealt with a single server working vacation queueing model with multiple types of server breakdowns. Wu and Yin [24] gave a detailed analysis on an $M/G/1$ retrial queue with non-exhaustive random vacations and unreliable server. Wang et al. [23] studied a discrete-time $Geo/G/1$ queue in which the server operates multiple vacations and may break down while working. Yang and Wu [25] investigated the N-policy $M/M/1$ queueing system with working vacation and server breakdowns. However, most of the papers are focused on queues where customers arrive at the system according to independent Poisson processes or geometrical arrival processes. Queueing systems with Lévy input (or Lévy-driven queues) are still not well investigated in literature. Recently, queueing systems with Lévy input have been attracting increasing attention in the applied probability and stochastic operations research communities. Lévy-driven queues covers the classical $M/G/1$ queue and the reflected Brownian motion as special cases. Kella and Boxma [16] first considered Lévy processes with secondary jump input which were applied to analyze queues with server vacations. Recently, Lieshout and Mandjes [20] analyzed tail asymptotics of a two-node tandem queue with spectrally positive Lévy input. Boxma et al. [5] analyzed a generic class of Lévy-driven queueing systems with server vacation. Dębicki and Mandjes [9] provided a survey on Lévy-driven queues. Dębicki et al. [8] focused on transient analysis of Lévy-driven tandem

queues. Palmowski et al. [21] considered a controlled fluid queueing model with Lévy input. Boxma and Kella [6] generalized known workload decomposition results for Lévy queues with secondary jump inputs and queues with server vacations or service interruptions. However, to the best of our knowledge, no work on Lévy queues with server breakdowns and vacations is found in the queueing literature.

In this paper, we consider a single-server Lévy-driven fluid queue with multiple vacations (exhaustive service) and a server subject to breakdowns and repairs which is motivated by the performance analysis of resources in communication networks. A breakdown at the server is represented by the arrival of a failure. The principal purpose of the present paper is to apply the martingale results which were derived in Kella and Whitt [17] to investigate the stochastic dynamics of the system and realize an extensive analysis of the system from the transient virtual waiting time process to the steady-state distribution of the waiting time process. In addition, we give a detailed analysis of the system busy period and the queue's correlation structure. Furthermore, we establish decomposition results for the model.

The rest of the paper is organized as follows. In Section 2, we introduce a few basic facts concerning Lévy processes and martingales. In Section 3, we describe the general model and formulate the model as a Lévy process modified to have random jumps at two classes of stopping times. In Section 4, we characterize the steady-state distribution of the virtual waiting time process and the mean length of the busy period. In Section 5, we address the transient distribution of the waiting time process as well as the queue's correlation structure. In Section 6, we present two stochastic decomposition results.

2. Preliminaries on Lévy processes

In this section, we mainly give several notations, definitions and propositions about Lévy processes. For more details, see [1, 2, 4, 22].

Denote $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}_+ = [0, \infty)$. Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_{t \geq 0}\})$ be a complete probability space endowed with a σ -field filtration $\mathcal{F}_{t \geq 0}$, i.e. an increasing family of sub-fields, which fulfils the usual conditions. That is, each \mathcal{F}_t is \mathbb{P} -complete and $\mathcal{F}_t = \cap_{s > t} \mathcal{F}_s$ for every t . Throughout, adapted, stopping times and martingales will be defined with respect to this filtration.

Definition 2.1. Let $X = (X_t, t \geq 0)$ be a stochastic process defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, if

- (i) $X_0 = 0$ (a.s.);
- (ii) X has independent and stationary increments;
- (iii) X is stochastically continuous, i.e. for all $a > 0$ and for all $s \geq 0$,

$$\lim_{t \rightarrow s} \mathbb{P}(|X_t - X_s| > a) = 0,$$

then X is a Lévy process.

Two central and key results in the foundations of Lévy processes are given by the following propositions (known as the Lévy-Khintchine formula and Lévy-Itô decomposition respectively).

Proposition 2.1. Let X be a Lévy process with Lévy measure ν ($\int_{\mathbb{R}-\{0\}} (|x|^2 \wedge 1) \nu(dx) < \infty$). Then

$$\mathbb{E} e^{i\theta X_t} = e^{t\psi(\theta)}, \quad t \geq 0,$$

where

$$\psi(\theta) = i\mu\theta - \frac{1}{2}\theta^2\sigma^2 + \int_{\mathbb{R}-\{0\}} (e^{i\theta x} - 1 - i\theta x \mathbf{1}_{|x|<1}) \nu(dx),$$

$\mu \in \mathbb{R}$ and $\sigma \geq 0$. Moreover, given ν, μ, σ^2 , the corresponding Lévy process is unique in distribution. Furthermore, the jump process of X is a Poisson point process with characteristic measure ν , that is

$$\nu(\Lambda) = \mathbb{E} \left[\sum_{0 < s \leq 1} \mathbf{1}_{\Lambda}(\Delta X_s) \right].$$

The function $\psi(\cdot)$ is called the Lévy exponent of the Lévy process.

Proposition 2.2. Let X be a Lévy process. Then X has a decomposition

$$X_t = B_t + \mu t + \int_{\{|x|<1\}} x(N_t(\cdot, dx) - t\nu(dx)) + \int_{\{|x|\geq 1\}} x N_t(\cdot, dx)$$

where B is a Brownian motion and for $0 \notin \bar{\Lambda}$, $N_t^\Lambda = \int_\Lambda N_t(\cdot, dx)$ is a Poisson process independent of B .

The following are some known fundamental results about Lévy processes:

- (i) Any Lévy process can be represented as an independent sum of a Brownian motion B and a ‘compound Poisson’-like process X^0 . If X has no negative jumps and the paths of X^0 are of bounded variation, then X^0 can be a subordinator which can be represented as a nonnegative compound Poisson process or as the limit of a sequence of nonnegative compound Poisson processes [3].
- (ii) If $\nu((-\infty, 0)) = 0$, then the Laplace-Stieltjes transform exists and is given by $\mathbb{E} \exp(-\theta X_t) = \exp(t\varphi(\theta))$ where

$$\varphi(\theta) = \log \mathbb{E} \exp\{-\theta X_1\} = -\mu\theta + \frac{1}{2}\theta^2\sigma^2 + \int_{\mathbb{R}_+} (e^{-\theta x} - 1 + \theta x \mathbf{1}_{|x|<1})\nu(dx),$$

here the function $\varphi(\cdot)$ is called the Laplace-Stieltjes exponent of the Lévy process. It is easy to get that $\varphi(0) = 0$ and $\varphi(\cdot)$ is convex; If X is not a subordinator, then $\varphi(\theta) \rightarrow \infty$ as $\theta \rightarrow \infty$; Whenever $EX_1 < 0$, $\varphi(\cdot)$ is strictly increasing. Throughout the paper, exponent will mean the Laplace-Stieltjes exponent $\varphi(\cdot)$ due to the convenience compared to the Lévy exponent $\psi(\cdot)$.

- (iii) If X has bounded jumps, then $\mathbb{E}|X_1|^n < \infty$ for every $n \geq 0$.

It is known that there always exists a version with sample paths of X that are càdlàg, which is therefore strong Markov. Throughout this paper, every Lévy process mentioned is assumed to be such.

In this paper, we focus only on the spectrally positive Lévy process, i.e. $\nu(-\infty, 0) = 0$. For any spectrally positive Lévy process X with $\beta > 0$ and $EX_1 < 0$. Denote by $T^\xi = \inf\{t | X_t = -\xi\}$ for any non-negative random variable ξ which is independent of X . Then it follows Theorem 3.12 in [18] that

$$E \exp\{-\beta T^\xi\} = \exp\{-\varphi^{-1}(\beta)\xi\}, \quad ET^\xi = \frac{E\xi}{-EX_1}.$$

We now give the generalized Pollaczek-Khinchine formula associated with a reflected Lévy process with no negative jumps which is also a celebrated formula in queueing theory. Given some random variable $0 \leq R_0 \in \mathcal{F}_0$, let

$$I_t = \left(- \inf_{0 \leq s \leq t} X_s - R_0 \right)^+, \quad R_t = R_0 + X_t + I_t, \quad t \geq 0.$$

Then R is a reflected Lévy process with I being its local time at zero. Since X has no negative jumps, I is continuous with $I_0 = 0$ and is the minimal right-continuous nondecreasing process such

that $R_t \geq 0$ for all t . The process $\{I_t, t \geq 0\}$ must only increase when $R_t = 0$, so that

$$\int_0^\infty \mathbf{1}_{\{R_t > 0\}} dI_t = 0.$$

The following result is the famous (generalized) Pollaczek-Khinchine formula in queueing theory. We can refer to the papers [16] and [9] for its proof.

Proposition 2.3. *If X is a spectrally positive Lévy process such that $EX_1 < 0$, then*

$$\lim_{t \rightarrow \infty} \mathbb{E} e^{-\theta R_t} = \frac{\theta \varphi'(0)}{\varphi(\theta)}, \quad \theta > 0.$$

We end this section by presenting the famous Kella-Whitt martingale associated with spectrally positive Lévy process X with exponent $\varphi(\theta)$ that we will apply it to analyze our fluid queue.

Proposition 2.4. *Let X be a spectrally positive Lévy process with exponent $\varphi(\theta)$. Let $L_t = \int_0^t dL_s^c + \sum_{0 \leq s \leq t} \Delta L_s$ be an adapted càdlàg process of bounded variation on finite intervals with continuous part $\{L_t^c\}$ and jumps $\Delta L_s = L_s - L_{s-}$, and define $Z_t = Z_0 + X_t + L_t$, where $Z_0 \in \mathcal{F}_0$. Then*

$$\begin{aligned} M_t = & \varphi(\theta) \int_0^t \exp\{-\theta Z_s\} ds + \exp\{-\theta Z_0\} - \exp\{-\theta Z_t\} - \theta \int_0^t \exp\{-\theta Z_s\} dY_s^c \\ & + \sum_{0 \leq s \leq t} [\exp\{-\theta Z_s\} - \exp\{-\theta Z_s - \Delta L_s\}] \end{aligned}$$

is a local martingale. In addition, if the expected variation of $\{L_t^c, t \geq 0\}$ and the expected number of jumps of $\{L_t, t \geq 0\}$ are finite on every finite interval and $\{Z_t, t \geq 0\}$ is a non-negative process (or bounded below), then $\{M_t, t \geq 0\}$ is a martingale.

The Kella-Whitt martingale is a stochastic integral with respect to the Wald martingale $N_t = \exp\{-\theta X_t - \varphi(\theta)t\}$ and the readers may refer to the paper [17] for its proof.

3. Model formulation

We consider a single fluid queue with a (nondecreasing) Lévy input and available-processing (or service) processes. For any $t \geq 0$, let X_t be the cumulative input of fluid over the interval $[0, t]$. We assume that X is a subordinator, that is, a Lévy process, having the exponent

$$\phi(\theta) = -a\theta + \int_{\mathbb{R}_+} (e^{-\theta x} - 1) \nu(dx),$$

where $a \geq 0$ and $\int_{\mathbb{R}_+} x\nu(dx) < \infty$. Let $S(t)$ be the cumulative available processing over the interval $[0, t]$. Here, we assume the server processes the fluid at a constant deterministic rate r (whenever the fluid level is positive). So that $S(t) = rt$, $t \geq 0$. We assume the storage space is unlimited. Let $\{\tau_n, n \geq 1\}$ be the strictly increasing sequence of stopping times at which the server fails. When the server fails, it is repaired immediately and the time required to repair it is a positive random variable ξ_n which is \mathcal{F}_{τ_n} -measurable for $n \geq 1$. When the system becomes empty, the server takes a vacation of random length η_n which is \mathcal{F}_{σ_n} -measurable for $n \geq 1$, where $\{\sigma_n, n \geq 1\}$ is a strictly increasing sequence of stopping times at which the server is leaving for a vacation. Vacations continue until, on return from a vacation, the server finds the system is non-empty.

Let W_t represent the virtual waiting time at time t . Denote the initial workload by $W_0 \in \mathcal{F}_0$. Then W_t can be defined by (the empty sum is zero)

$$W_t = W_0 + X_t - rt + \sum_{i=1}^{N_t^R} \xi_i + \sum_{i=1}^{N_t^V} \eta_i, \quad t \geq 0, \quad (3.1)$$

where

$$N_t^R = \sup\{n | \tau_n \leq t\}, \quad N_t^V = \sup\{n | \sigma_n \leq t\},$$

$$\sigma_n = \inf \left\{ t \geq 0 \left| W_0 + X_t - rt + \sum_{i=1}^{N_t^R} \xi_i + \sum_{i=1}^{n-1} \eta_i = 0 \right. \right\}, \quad n \geq 1.$$

Remark 3.1. *It should be noted that here W_t , $t \geq 0$, is not the workload process. However, if ξ_n represents the total arriving workload during the n th repair period and η_n corresponds to the total arriving workload during the n th vacation period, then W_t is the workload in the system conditioned on being in the active periods which are defined as the normal processing fluid periods (excluding the repair periods and vacation periods).*

Define $Y_t = X_t - rt$. Then Y is a spectrally positive Lévy process with exponent $\varphi(\theta) = \phi(\theta) + r\theta$. Denote $\rho = -\phi'(0)$ and impose the conditions $\rho < r$ and $\mathbb{E}|Y_t| < \infty$ for all $t \geq 0$.

4. The steady-state distribution

In this section, we characterize the limiting distribution of the virtual waiting time W_t . Let \xrightarrow{d} denote convergence in distribution and let \xrightarrow{p} denote convergence in probability. Under the condition $\rho < r$, we suppose $W_t \xrightarrow{d} W$ and $t^{-1}\mathbb{E}W_t \rightarrow 0$, as $t \rightarrow \infty$, where W is a random variable.

In order to derive the steady-state distribution of the virtual waiting time W_t , we give the following lemmas.

Lemma 4.1. *If $E(N_t^R + N_t^V) < \infty$ for all t , then $\{M_t, t \geq 0\}$ is a zero-mean real-valued martingale with respect to $\{\mathcal{F}_t, t \geq 0\}$, where*

$$M_t = \varphi(\theta) \int_0^t e^{-\theta W_s} ds + e^{-\theta W_0} - e^{-\theta W_t} - \sum_{k=1}^{N_t^R} \left[e^{-\theta(W_{\tau_k} - \xi_k)} - e^{-\theta W_{\tau_k}} \right] - \sum_{k=1}^{N_t^V} (1 - e^{-\theta \eta_k}). \quad (4.1)$$

Proof. Let $H_t = \sum_{i=1}^{N_t^R} \xi_i + \sum_{i=1}^{N_t^V} \eta_i$. Then $W_t = W_0 + Y_t + H_t$ and $\{H_t, t \geq 0\}$ is adapted of bounded variation on every finite interval, and $H_t^c = 0$ for all $t \geq 0$. Since $E(N_t^R + N_t^V) < \infty$, by Proposition 2.4 and through a trivial manipulation, we get that (4.1) is a martingale. \square

Lemma 4.2. *Let T is a nonnegative random variable with possible integer values $0, 1, 2, \dots$ and $\mathbb{E}T < \infty$. If $\{\zeta_n, n \geq 1\}$ is an i.i.d. sequence with ζ_n independent of $\mathbf{1}_{\{T \geq n\}}$ and $\mathbb{E}\zeta_1 < \infty$, then*

$$\mathbb{E} \sum_{n=1}^T \zeta_n = (\mathbb{E}\zeta_1)(\mathbb{E}T).$$

Proof. Since ζ_n is independent of $\mathbf{1}_{\{T \geq n\}}$, the bounded convergence theorem yields

$$\begin{aligned} \mathbb{E} \left[\sum_{n=1}^T \zeta_n \right] &= \mathbb{E} \left[\sum_{n=1}^{\infty} \zeta_n \mathbf{1}_{\{T \geq n\}} \right] = \sum_{n=1}^{\infty} (\mathbb{E}\zeta_n)(\mathbb{E}\mathbf{1}_{\{T \geq n\}}) = \mathbb{E}\zeta_1 \sum_{n=1}^{\infty} (\mathbb{E}\mathbf{1}_{\{T \geq n\}}) \\ &= \mathbb{E}\zeta_1 \sum_{n=1}^{\infty} \mathbb{P}(T \geq n) = (\mathbb{E}\zeta_1)(\mathbb{E}T). \end{aligned}$$

\square

With the help of the above two lemmas, we can derive the limiting distribution of the virtual waiting time provided in the following theorem which is the most important result of this paper.

Theorem 4.1. *Let $\{\xi_n, n \geq 1\}$ and $\{\eta_n, n \geq 1\}$ be two positive i.i.d. sequences with $\mathbb{E}\xi_1 < \infty$ and $\mathbb{E}\eta_1 < \infty$, respectively. Suppose that $n^{-1}\tau_n \xrightarrow{p} \lambda_R^{-1}$ and $n^{-1}\sigma_n \xrightarrow{p} \lambda_V^{-1}$ as $n \rightarrow \infty$ for $0 < \lambda_R^{-1} < \infty$ and $0 < \lambda_V^{-1} < \infty$, respectively. Further, assume that $\{W_{\tau_k}, k \geq 1\}$ and $\{W_{\tau_k} - \xi_k, k \geq 1\}$ are stationary and ergodic that $(W_{\tau_k} - \xi_k, W_{\tau_k}) \xrightarrow{d} (W^-, W^+)$ where W^-, W^+ are two proper random variables. If ξ_n is independent of $\mathbf{1}_{\{N_1^R \geq n\}}$ and η_n is independent of $\mathbf{1}_{\{N_1^V \geq n\}}$, then for $\theta > 0$,*

$$\lambda_R = \frac{p\varphi'(0)}{\mathbb{E}\xi_1}, \quad \lambda_V = \frac{(1-p)\varphi'(0)}{\mathbb{E}\eta_1}, \quad (4.2)$$

$$\lim_{t \rightarrow \infty} t^{-1} \mathbb{E} \sum_{k=1}^{N_t^R} \xi_k = -p \mathbb{E} Y_1, \quad \lim_{t \rightarrow \infty} t^{-1} \mathbb{E} \sum_{k=1}^{N_t^R} \eta_k = -(1-p) \mathbb{E} Y_1, \quad (4.3)$$

$$\lim_{t \rightarrow \infty} \mathbb{E} e^{-\theta W_t} = \mathbb{E} e^{-\theta W} = \frac{\theta \varphi'(0)}{\varphi(\theta)} \left[p \frac{\mathbb{E} e^{-\theta W^-} - \mathbb{E} e^{-\theta W^+}}{\theta \mathbb{E} \xi_1} + (1-p) \frac{1 - \mathbb{E} e^{-\theta \eta_1}}{\theta \mathbb{E} \eta_1} \right], \quad (4.4)$$

where $0 \leq p \leq 1$.

Proof. Dividing (3.1) by t and taking expectation, we get

$$t^{-1} \mathbb{E} W_t = t^{-1} \mathbb{E} W_1 + \mathbb{E} Y_1 + t^{-1} \mathbb{E} \sum_{k=1}^{N_t^R} \xi_k + t^{-1} \mathbb{E} \sum_{k=1}^{N_t^V} \eta_k. \quad (4.5)$$

where $t^{-1} \mathbb{E} W_t \rightarrow 0$ by the assumption. Thus, we have proved (4.3).

Taking $W_0 = W^*$ where W^* is an independent copy of W , $\{W_t, t \geq 0\}$ becomes stationary. By Lemma 4.2 and (3.1), we get

$$\mathbb{E} N_1^R \mathbb{E} \xi_1 + \mathbb{E} N_1^V \mathbb{E} \eta_1 = -\mathbb{E} Y_1. \quad (4.6)$$

So that,

$$\mathbb{E} N_1^R \mathbb{E} \xi_1 = -p \mathbb{E} Y_1, \quad \mathbb{E} N_1^V \mathbb{E} \eta_1 = -(1-p) \mathbb{E} Y_1.$$

Hence,

$$\mathbb{E} N_1^R = \frac{p \varphi'(0)}{\mathbb{E} \xi_1}, \quad \mathbb{E} N_1^V = \frac{(1-p) \varphi'(0)}{\mathbb{E} \eta_1}. \quad (4.7)$$

Conditions $n^{-1} \tau_n \xrightarrow{p} \lambda_R^{-1}$ and $n^{-1} \sigma_n \xrightarrow{p} \lambda_V^{-1}$ as $n \rightarrow \infty$ imply that $t^{-1} N_t^R \xrightarrow{p} \lambda_R$ and $t^{-1} N_t^V \xrightarrow{p} \lambda_V$. Together with $t^{-1} N_t^R \rightarrow \mathbb{E} N_1^R$ and $t^{-1} N_t^V \rightarrow \mathbb{E} N_1^V$ w.p.1 as $t \rightarrow \infty$, this implies (4.2).

From Lemma 4.2, optional stopping at $t = 1$ in (4.1) yields

$$0 = \varphi(\theta) \mathbb{E} e^{-\theta W} - \mathbb{E} N_1^R \left[\mathbb{E} e^{-\theta W^-} - \mathbb{E} e^{-\theta W^+} \right] - \mathbb{E} N_1^V (1 - \mathbb{E} e^{-\theta \eta_1}) \quad (4.8)$$

Substituting (4.7) into (4.8), we obtain (4.4).

□

Remark 4.1. From [7], we have $\varphi(\theta) = 0$ for some $\theta > 0$ if and only if Y_t has a lattice distribution. But for a Lévy process with no negative jumps and $\mathbb{E} Y_t < 0$, it is not possible. In addition, when

$N_t^R \equiv 0$ for all t , i.e., $p = 0$, the present system reduces to the Lévy process-driven queue with server vacations. It may be noted that the equation (4.4) after putting $p = 0$ agrees with the equation (4.6) presented in Kella and Whitt [16]. We should note that when $N_t^V \equiv 0$ for all t , the present system reduces to the Lévy process-driven queue with server breakdowns. However, the equation (4.4) in Theorem 4.1 after putting $p = 1$ cannot characterize the limiting distribution of Lévy-driven queue with server breakdowns due to the fact that $\{W_t, t \geq 0\}$ is not a non-negative process.

Corollary 4.1. *When the system is stable, we have*

$$\begin{aligned}\omega = \mathbb{E}W &= \frac{\varphi''(0)}{2\varphi'(0)} + p \frac{\mathbb{E}(W^+)^2 - \mathbb{E}(W^-)^2}{2\mathbb{E}\xi_1} + (1-p) \frac{\mathbb{E}\eta_1^2}{2\mathbb{E}\eta_1}, \\ v = \mathbb{V}ar W &= \frac{1}{3} \frac{\varphi'''(0)}{\varphi'(0)} - \frac{1}{4} \left(\frac{\varphi''(0)}{\varphi'(0)} \right)^2 + (1-p) \left[\frac{1}{3} \frac{\mathbb{E}\eta_1^3}{\mathbb{E}\eta_1} - \frac{1}{4} \left(\frac{\mathbb{E}\eta_1^2}{\mathbb{E}\eta_1} \right)^2 \right] \\ &+ p \left[\frac{1}{3} \frac{\mathbb{E}(W^+)^3 - \mathbb{E}(W^-)^3}{\mathbb{E}\xi_1} - \frac{1}{4} \left(\frac{\mathbb{E}(W^+)^2 - \mathbb{E}(W^-)^2}{\mathbb{E}\xi_1} \right)^2 \right].\end{aligned}$$

Next, we analyze the system busy period which is defined as the time period which starts at the epoch when a vacation is completed and the server begins processing the fluid and ends at the next epoch when the system is empty. Note that a busy period includes the normal processing time of the fluid in the system and some possible repair times of the server due to the server breakdowns.

Lemma 4.3. *Define $T_n = \inf\{t | Y_t + \sum_{k=1}^{N_t^R} \xi_k + \sum_{i=1}^n B_i = 0\}$, $n \geq 1$, where $\{B_i, i \geq 1\}$ is a positive i.i.d. sequence with $\mathbb{E}B_1 < \infty$. If $0 \leq p < 1$, then*

$$\mathbb{E}T_n = \frac{n\mathbb{E}B_1}{(1-p)\varphi'(0)}. \quad (4.9)$$

Proof. Let $T_n^b = \inf\{t | Y_t + \sum_{k=1}^{N_t^R} \xi_k + nb = 0\}$, $b > 0$, $n \geq 1$. Since Y is a spectrally positive Lévy process, $\{T_n^b, n \geq 1\}$ is a random walk with $0 < T_n^b < T_{n+1}^b$ w.p.1. Thus, $\mathbb{E}T_n^b = n\mathbb{E}T_1^b$. Since $\mathbb{E}Y_t < 0$ and $0 \leq p < 1$, $\mathbb{E}(Y_t + \sum_{k=1}^{N_t^R} \xi_k) = -(1-p)\varphi'(0)t < 0$. By Theorem 8.4.4 of Chung (1974), we have $\mathbb{E}T_1^b < \infty$. Since $\mathbb{E}|Y_t| < \infty$, $t^{-1}Y_t \rightarrow \mathbb{E}Y_1$ w.p.1 as $t \rightarrow \infty$ by the Kolmogorov strong law of large numbers. Taking into account the fact that $Y_{T_n^b} + \sum_{k=1}^{N_{T_n^b}^R} \xi_k = -nb$, we get

$$\begin{aligned}-b &= n^{-1} \left(Y_{T_n^b} + \sum_{k=1}^{N_{T_n^b}^R} \xi_k \right) = n^{-1} T_n^b \left[(T_n^b)^{-1} Y_{T_n^b} + (T_n^b)^{-1} \sum_{k=1}^{N_{T_n^b}^R} \xi_k \right] \\ &\longrightarrow \mathbb{E}T_1^b [\mathbb{E}Y_1 + (\mathbb{E}N_1^R)(\mathbb{E}\xi_1)] = \mathbb{E}T_1^b [-(1-p)\varphi'(0)] \quad \text{w.p.1 as } n \rightarrow \infty.\end{aligned}$$

So that $\mathbb{E}T_n^b = \frac{nb}{(1-p)\varphi'(0)}$. Finally, by conditioning and unconditioning, we obtain (4.9). \square

By the argument of Lemma 4.3, we obtain the following result.

Theorem 4.2. *Let $T = \inf\{t|W_t = 0\}$, where W_0 is distributed according to the stationary distribution. Then the mean length of the busy period is given by*

$$\mathbb{E}T = \frac{\mathbb{E}W}{(1-p)\varphi'(0)}.$$

Remark 4.2. *Note that for the $M/G/1$ queue with server breakdowns and multiple vacations, if B_i is distributed as a service time, then T_n is the n -order busy period which is defined as the time period which starts when a vacation is completed and there are n customers in the system and ends at the next departure epoch when the system is empty.*

5. The transient distribution

In this section, we focus on analyzing the transient distribution of W_t in terms of Laplace-Stieltjes transform, for some $t > 0$, conditional on $W_0 = x$.

Theorem 5.1. *Let T be exponentially distributed with mean $1/\gamma$, independently of Y , τ_n and σ_n , $n \geq 1$. For $\theta > 0$ and $x \geq 0$,*

$$\begin{aligned} \mathbb{E}_x e^{-\theta W_T} &= \frac{\gamma}{\varphi(\theta) - \gamma} e^{-\varphi^{-1}(\gamma)x} \\ &\times \left\{ p \frac{\mathbb{E}e^{-\theta W^-} - \mathbb{E}e^{-\theta W^+}}{\mathbb{E}e^{-\varphi^{-1}(\gamma)W^-} - \mathbb{E}e^{-\varphi^{-1}(\gamma)W^+}} + (1-p) \frac{1 - \mathbb{E}e^{-\theta \eta_1}}{1 - \mathbb{E}e^{-\varphi^{-1}(\gamma)\eta_1}} - e^{-[\theta - \varphi^{-1}(\gamma)]x} \right\}. \end{aligned} \quad (5.1)$$

Proof. By Lemma 4.1, we have

$$\begin{aligned} 0 &= \mathbb{E}M_T = \varphi(\theta) \int_0^\infty \int_0^t \gamma e^{-\gamma t} e^{-\theta W_s} ds dt + e^{-\theta x} - \mathbb{E}_x e^{-\theta W_T} \\ &\quad - \mathbb{E} \sum_{k=1}^{N_T^R} \left[e^{-\theta(W_{\tau_k} - \xi_k)} - e^{-\theta W_{\tau_k}} \right] - \mathbb{E} \sum_{k=1}^{N_T^V} (1 - e^{-\theta \eta_k}) \\ &= \frac{\varphi(\theta)}{\gamma} \mathbb{E}_x e^{-\theta W_T} + e^{-\theta x} - \mathbb{E}_x e^{-\theta W_T} - \mathbb{E}N_T^R \left[\mathbb{E}e^{-\theta W^-} - \mathbb{E}e^{-\theta W^+} \right] \\ &\quad - \mathbb{E}N_T^V (1 - \mathbb{E}e^{-\theta \eta_1}). \end{aligned}$$

So that

$$\mathbb{E}_x e^{-\theta W_T} = \frac{\gamma}{\varphi(\theta) - \gamma} \left\{ \mathbb{E}N_T^R \left[\mathbb{E}e^{-\theta W^-} - \mathbb{E}e^{-\theta W^+} \right] + \mathbb{E}N_T^V (1 - \mathbb{E}e^{-\theta \eta_1}) - e^{-\theta x} \right\}.$$

Note that when $\mathbb{E}Y_1 < 0$, $\varphi(\theta)$ is increasing on $[0, \infty)$. Therefore, the inverse of $\varphi(\theta)$ is well defined on $[0, \infty)$. Hence, the equation $\varphi(\theta) = \gamma$ has exactly one root $\theta = \varphi^{-1}(\gamma)$ on $[0, \infty)$. Finally, using the fact that the root of the denominator should be a root of the numerator as well, we can obtain (5.1). \square

Next, we derive explicitly the Laplace transform corresponding to the correlation of the virtual waiting time process:

$$c(t) = \frac{\text{Cov}(W_0, W_t)}{\sqrt{\text{Var}W_0 \cdot \text{Var}W_t}} = \frac{\mathbb{E}(W_0 W_t) - (\mathbb{E}W_0)^2}{\text{Var}W_0} = \frac{\mathbb{E}(W_0 W_t) - \omega^2}{v}.$$

Here, we assume the system is in steady-state at time 0.

Theorem 5.2. For $\theta > 0$ and ω, v as in Corollary 4.1,

$$\begin{aligned} \int_0^\infty c(t)e^{-\theta t} dt &= \frac{1}{\theta} - \frac{\omega\varphi'(0)}{v\theta^2} + \frac{\mathbb{E}(W e^{-\varphi^{-1}(\theta)W})}{v\theta} \\ &\quad \times \left\{ \frac{p\mathbb{E}\xi_1}{\mathbb{E}e^{-\varphi^{-1}(\theta)W^-} - \mathbb{E}e^{-\varphi^{-1}(\theta)W^+}} + \frac{(1-p)\mathbb{E}\eta_1}{1 - \mathbb{E}e^{-\varphi^{-1}(\theta)\eta_1}} \right\}, \end{aligned}$$

where

$$\begin{aligned} \mathbb{E}(W e^{-\varphi^{-1}(\theta)W}) &= \left[\frac{\varphi'(0)}{\varphi(\theta)} - \frac{\theta\varphi'(0)\varphi'(\theta)}{\varphi^2(\theta)} \right] \left[p \frac{\mathbb{E}e^{-\theta W^-} - \mathbb{E}e^{-\theta W^+}}{\theta\mathbb{E}\xi_1} + (1-p) \frac{1 - \mathbb{E}e^{-\theta\eta_1}}{\theta\mathbb{E}\eta_1} \right] \\ &\quad + \frac{\varphi'(0)}{\varphi(\theta)} \frac{p}{\mathbb{E}\xi_1} \left[\mathbb{E}(W^+ e^{-\theta W^+}) - \mathbb{E}(W^- e^{-\theta W^-}) - \frac{\mathbb{E}e^{-\theta W^-} - \mathbb{E}e^{-\theta W^+}}{\theta} \right] \\ &\quad + \frac{\varphi'(0)}{\varphi(\theta)} \frac{1-p}{\mathbb{E}\eta_1} \left[\mathbb{E}(\eta_1 e^{-\theta\eta_1}) - \frac{1 - \mathbb{E}e^{-\theta\eta_1}}{\theta} \right]. \end{aligned}$$

Proof. Let T be exponentially distributed with mean $1/\theta$. From (5.1), we get

$$\int_0^\infty \theta e^{-\theta t} \mathbb{E}_x W_t dt = -\frac{\varphi'(0)}{\theta} + e^{-\varphi^{-1}(\theta)x} \left[\frac{p\mathbb{E}\xi_1}{\mathbb{E}e^{-\varphi^{-1}(\theta)W^-} - \mathbb{E}e^{-\varphi^{-1}(\theta)W^+}} + \frac{(1-p)\mathbb{E}\eta_1}{1 - \mathbb{E}e^{-\varphi^{-1}(\theta)\eta_1}} \right] + x. \quad (5.2)$$

Straightforward calculus yields

$$\begin{aligned} \int_0^\infty c(t)e^{-\theta t} dt &= \frac{1}{v} \int_0^\infty [\mathbb{E}(W_0 W_t) - \omega^2] e^{-\theta t} dt \\ &= \frac{1}{v} \int_0^\infty \int_0^\infty x \mathbb{E}_x W_t e^{-\theta t} d\mathbb{P}(W_0 \leq x) dt - \frac{\omega^2}{v\theta}. \end{aligned} \quad (5.3)$$

Substituting (5.2) into (5.3), we obtain

$$\int_0^\infty c(t)e^{-\theta t} dt = -\frac{\omega^2}{v\theta} +$$

$$\begin{aligned}
& \int_0^\infty \frac{x}{v\theta} \left\{ -\frac{\varphi'(0)}{\theta} + e^{-\varphi^{-1}(\theta)x} \left[\frac{p\mathbb{E}\xi_1}{\mathbb{E}e^{-\varphi^{-1}(\theta)W^-} - \mathbb{E}e^{-\varphi^{-1}(\theta)W^+}} + \frac{(1-p)\mathbb{E}\eta_1}{1 - \mathbb{E}e^{-\varphi^{-1}(\theta)\eta_1}} \right] + x \right\} d\mathbb{P}(W_0 \leq x) \\
&= \frac{1}{\theta} - \frac{\omega\varphi'(0)}{v\theta^2} + \frac{\mathbb{E}(We^{-\varphi^{-1}(\theta)W})}{v\theta} \left\{ \frac{p\mathbb{E}\xi_1}{\mathbb{E}e^{-\varphi^{-1}(\theta)W^-} - \mathbb{E}e^{-\varphi^{-1}(\theta)W^+}} + \frac{(1-p)\mathbb{E}\eta_1}{1 - \mathbb{E}e^{-\varphi^{-1}(\theta)\eta_1}} \right\},
\end{aligned}$$

where $\mathbb{E}(We^{-\varphi^{-1}(\theta)W})$ can be obtained by differentiating (4.4). This completes the proof. \square

6. Stochastic decompositions

In this section, we identify some stochastic decomposition properties of the virtual waiting time for our fluid queueing system. Stochastic decomposition properties have been studied in many vacation models. The classical stochastic decomposition property shows that the steady-state system size at an arbitrary point can be represented as the sum of two independent random variable, one of which is the system size of the corresponding standard queueing system without server vacations and the other random variable depends on the meaning of vacations in specific cases (see Fuhrmann and Cooper [11] and Doshi [10]). In addition, stochastic decomposition properties have also been held for Lévy-driven queues with interruptions as well as càdlàg processes with certain secondary jump inputs (see Kella and Whitt [16] and Ivanovs and Kella [13]). It's worth mentioning that, recently, Boxma and Kella [6] generalize known workload decomposition results for Lévy queues with secondary jump inputs and queues with server vacations or service interruptions. In particular, in the context of our fluid queueing system, we have the following decomposition results.

Lemma 6.1. *There are two random variables U and V such that*

$$\frac{\mathbb{E}e^{-\theta W^-} - \mathbb{E}e^{-\theta W^+}}{\theta\mathbb{E}\xi_1} = \mathbb{E}e^{-\theta U}, \quad (6.1)$$

$$\frac{1 - \mathbb{E}e^{-\theta\eta_1}}{\theta\mathbb{E}\eta_1} = \mathbb{E}e^{-\theta V}. \quad (6.2)$$

Proof. The left hand side of (6.1) is the Laplace transformation of the function

$$f(x) = \frac{\mathbb{P}(W^+ > x) - \mathbb{P}(W^- > x)}{\mathbb{E}\xi_1},$$

which is a bona fide probability density function.

The left hand side of (6.2) is the Laplace transformation of the function

$$g(x) = \frac{\mathbb{P}(\eta_1 > x)}{\mathbb{E}\eta_1},$$

which is a bona fide probability density function (stationary residual life density of η_1). \square

Applying Lemma 6.1, we can get the following theorem to characterize the stochastic decomposition property.

Theorem 6.1. *Under the conditions of Theorem 4.1, the distribution of W is the convolution of two distributions one of which is the distribution of R in Section 2 and the other is the mixture of the distribution of U with probability p and the stationary residual life distribution of η_1 with probability $1 - p$.*

Furthermore, under some conditions, we have another stochastic decomposition result.

Lemma 6.2. *Under the conditions of Theorem 4.1, if positive random variables ξ_n , $n \geq 1$ are independent of W_0 , Y_{τ_n} and $\sum_{i=1}^{N_{\tau_n}^V} \eta_i$, then we have*

$$\mathbb{E}e^{-\theta W} = \frac{\theta\varphi'(0)}{\varphi(\theta)} \left[p \frac{1 - \mathbb{E}e^{-\theta\xi_1}}{\theta\mathbb{E}\xi_1} \mathbb{E}e^{-\theta W^-} + (1 - p) \frac{1 - \mathbb{E}e^{-\theta\eta_1}}{\theta\mathbb{E}\eta_1} \right]. \quad (6.3)$$

Proof. Since ξ_k is independent of $W_{\tau_k} - \xi_k = W_0 + Y_{\tau_n} + \sum_{i=1}^{k-1} \xi_i + \sum_{i=1}^{N_{\tau_k}^V} \eta_i$,

$$\begin{aligned} \frac{\mathbb{E}e^{-\theta(W_{\tau_1} - \xi_1)} - \mathbb{E}e^{-\theta W_{\tau_1}}}{\theta\mathbb{E}\xi_1} &= \frac{\mathbb{E}e^{-\theta(W_{\tau_1} - \xi_1)}(1 - e^{-\theta\xi_1})}{\theta\mathbb{E}\xi_1} \\ &= \mathbb{E}e^{-\theta(W_{\tau_1} - \xi_1)} \frac{1 - \mathbb{E}e^{-\theta\xi_1}}{\theta\mathbb{E}\xi_1} \\ &= \mathbb{E}e^{-\theta W^-} \frac{1 - \mathbb{E}e^{-\theta\xi_1}}{\theta\mathbb{E}\xi_1}. \end{aligned} \quad (6.4)$$

Substituting (6.4) into (4.4) yields (6.3). \square

From Lemma 6.2, we obtain the following theorem which provides another stochastic decomposition.

Theorem 6.2. *If the assumptions of Lemma 6.2 are satisfied, then the distribution of U is the convolution of the distribution of W^- and the stationary residual life distribution of ξ_1 .*

References

- [1] APPLEBAUM, D. (2004). *Lévy Processes and Stochastic Calculus*. Cambridge: Cambridge University Press.
- [2] ASMUSSEN, S. (2003). *Applied Probability and Queues*. New York: Springer.
- [3] ASMUSSEN, S. AND ALBRECHER, H. (2010). *Ruin Probabilities*. Singapore: World Scientific.
- [4] BERTOIN, J. (1998). *Lévy Processes*. Cambridge: Cambridge University Press.
- [5] BOXMA, O. AND KELLA, O. (2010). On a generic class of lévy-driven vacation models. *Probability in the Engineering and Informational Sciences* **24**, 1–12.
- [6] BOXMA, O. AND KELLA, O. (2014). Decomposition results for stochastic storage processes and queues with alternating lévy inputs. *Queueing Systems* **77**, 97–112.
- [7] CHUNG, K. L. (1998). *A Course in Probability Theory*. New York: Academic.
- [8] DEBICKI, K., MANDJES, M. AND SIERPINSKA-TULACZ, I. (2013). Transient analysis of lévy-driven tandem queues. *Statistics and Probability Letters* **83**, 1776–1781.
- [9] DEBICKI, K. AND MANDJES, M. (2012). Lévy-driven queues. *Surveys in Operations Research and Management Science* **17**, 15–37.
- [10] DOSHI, B. T. (1990). Conditional and unconditional distributions for M/G/1 type queues with server vacations. *Queueing Systems* **7**, 229–252.
- [11] FUHRMANN, S. M. AND COOPER, R. B. (1985). Stochastic decomposition in the M/G/1 queue with generalized vacations. *Operations Research* **33**, 1117–1129.
- [12] GRAY, W. J., WANG, P. P. AND SCOTT, M. (2000). A vacation queueing model with service breakdowns. *Applied Mathematical Modelling* **24**, 391–400.
- [13] IVANOV, J. (2013). Another look into decomposition results. *Queueing Systems* **75**, 19–28.
- [14] JAIN, M. AND JAIN, A. (2010). Working vacations queueing model with multiple types. *Applied Mathematical Modelling* **34**, 1–13.

- [15] KE, J. C. (2003). Optimal strategy policy in batch arrival queue with server breakdowns and multiple vacations. *Mathematical Methods of Operations Research* **58**, 41–56.
- [16] KELLA, O. AND WHITT, W. (1991). Queues with server vacations and lévy processes with secondary jump input. *Annals of Applied Probability* **1**, 104–117.
- [17] KELLA, O. AND WHITT, W. (1992). Useful martingales for stochastic processes with levy input. *Journal of Applied Probability* **29**, 396–403.
- [18] KYPRIANOU, A. (2006). *Introductory Lectures on Fluctuations of Lévy Processes with Applications*. Berlin: Springer.
- [19] LI, W., SHI, D. AND X., C. (1997). Reliability analysis of M/G/1 queueing systems with server breakdowns and vacations. *Journal of Applied Probability* **34**, 546–555.
- [20] LIESHOUT, P. AND MANDJES, M. (2008). Asymptotic analysis of lévy-driven tandem queues. *Queueing Systems* **60**, 203–226.
- [21] PALMOWSKI, Z., VLASIOU, M. AND ZWART, B. (2014). A lévy input fluid queue with input and workload regulation. *Queueing Systems* **76**, 21–36.
- [22] SATO, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge: Cambridge University Press.
- [23] WANG, T. Y., KE, J. C. AND M., C. F. (2013). Analysis of a discrete-time queue with server subject to vacations and breakdowns. *Journal of Industrial and Production* **30**, 54–66.
- [24] WU, J. AND YIN, X. (2011). An M/G/1 retrial G-queue with non-exhaustive random vacations and unreliable server. *Computers and Mathematics with Applications* **62**, 2314–2329.
- [25] YANG, D. Y. AND WU, C. H. (2015). Cost-minimization analysis of a working vacation queue with n-policy and server breakdowns. *Computers & Industrial Engineering* **82**, 151–158.